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SHELL FINITE ELEMENT METHOD VIA REISSNER'S PRINCIPLE

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Abstract—A specialized form of Reissner's variational principle for stresses and displacements suitable to formulate a finite element version of the governing equations for thin shells is presented. The Euler equations of the modified Reissner's functional are three force equilibrium equations and three moment curvature relations expressed in terms of the linear displacement vector and stress couple vectors. The boundary conditions are derived as natural conditions of this functional. The shell domain is divided into a set of triangular or quadrilateral elements for the purpose of defining a set of piecewise linear functions as comparison functions for the dependent variables of this formulation. The use of these functions, expressed in terms of their values at the nodes, in the variational principle results in a set of algebraic simultaneous equations for the nodal variables. A characteristic of the method is its algebraic simplicity and accuracy when compared with other approximate methods of solution. Some numerical examples practically illustrate these characteristics of the method. When this method is specialized for plate bending analysis a significant improvement with respect to a similar previously known finite element procedure is observed.

NOTATION

a_i, b_i	Cartesian components of the oriented side "i" of the projection in the x_1x_2 plane (Fig. 2)
A	area of the projection of an element into the x_1x_2 plane
A_i	equation (33) and Fig. 2
C_s	coefficient that relates the transverse shear force with transverse shear strains (equation (19))
C_{ij}	coupling coefficients in the linear coefficient matrix $\mathbf{L}_{ij}$ (equation (48))
D	portion of the boundary with displacement boundary conditions
E	Youngs modulus
F_{ij}	flexibility coefficients in equation (48)
h	shell thickness
I	Reissner's functional
i, j, k	nodal points in a triangular element taken in a counterclockwise sense (Fig. 2)
$\mathbf{L}$	linear coefficient matrix (equation (47))
$\mathbf{M}_i$	stress couples
M_{ij}	stress couple components (equation (16))
$\mathbf{M}_n$	stress couple at the boundary (equation (81))
$\mathbf{N}_i$	stress resultant
N_{ij}, Q_i	stress resultant components (equation (13))
$\mathbf{N}_n$	stress resultant at the boundary (equation (79))
$\mathbf{p}$	load intensity vector with components (p_1, p_2, p_n)
$\mathbf{P}$	vector of right hand sides in equation (46)
$\mathbf{r}$	position vector (Fig. 1)
R_{ij}	radii of curvature of shell middle surface
S	portion of the boundary with stress boundary conditions
S_{ij}	stiffness coefficients in equation (48)
$\mathbf{u}$	linear displacement vector with components u, v, and w (equation (29))
$\mathbf{u}_n$	displacement vector at the boundary with components v_s, v_n and w (equation (78))
U	potential energy density of the applied surface loads
$\mathbf{V}$	vector of unknowns in equations (46)
W	strain energy density in terms of stresses

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W_M	strain energy density of in-plane stress resultants in terms of $\mathbf{u}$
W_B	strain energy density of bending stresses in terms of $\mathbf{M}_i$
α_i	Lamé parameters: $\alpha_j = \left\lvert \frac{\partial \mathbf{r}}{\partial x_j} \right\rvert$
γ_i	transverse shear strains
Λ	angle of obliqueness of shell coordinate system
$\boldsymbol{\varepsilon}_i$	linear strain vector with components ε_{ij} and γ_i (equation 15)
$\boldsymbol{\phi}$	rotation vector with components β_1, β_2, and ω (equation 18)
$\boldsymbol{\phi}_n$	rotation vector at the boundary
$\boldsymbol{\kappa}_i$	angular strain vector with components κ_{ij} and χ_i (equation (14))
ν	poisson ratio
$(\ )_{,i}$	denotes differentiation with respect to x_i
ζ_i	triangular coordinates (equation (32))
x_1, x_2, z	shell coordinate system
$\mathbf{t}_1, \mathbf{t}_2, \mathbf{n}$	unit vectors along the coordinate lines x_1, x_2 and along the normal to the shell respectively
$\mathbf{t}_s, \mathbf{t}_n, \mathbf{n}$	unit vectors at the oriented boundary
s	curvilinear coordinate at the boundary

1. INTRODUCTION

THE governing equations of the theory of thin shells can be conveniently derived by means of variational considerations, particularly when an approximate solution of a shell problem is desired. Reissner's Principle, [1–3, 5] specialized for thin shell theory from its three dimensional elasticity formulation, provides in many practical applications a sufficiently general variational statement of the shell equations.

The present work is concerned with a variational formulation of the shell equations and its solution by means of finite element technique whereby any variable can be expressed in terms of at most the first derivatives of those variables which are solved for explicitly. The desired variational statement is obtained by suitable specialization of Reissner's Principle. The desirability of such a method for shell analysis is twofold—on one hand it avoids the consideration of derivatives higher than the first. In that way, terms which tend to introduce large numerical errors need not be considered; on the other hand, it allows the use of piecewise linear functions as comparison functions and thus results in discrete equations which are algebraicly simple.

The present work deals only with the equations of the linear theory of thin shells, but a similar treatment of the nonlinear theory of shallow shells and linearized stability analysis has also been developed [11].

2. GOVERNING EQUATIONS

Reissner's Principle states that

$$\delta I = 0 \tag{1}$$

where

$$I = \iint (\mathbf{N}_1 \cdot \boldsymbol{\varepsilon}_1 + \mathbf{N}_2 \cdot \boldsymbol{\varepsilon}_2 + \mathbf{M}_1 \cdot \boldsymbol{\kappa}_1 + \mathbf{M}_2 \cdot \boldsymbol{\kappa}_2 - W + U)\,\mathrm{d}A$$
$$- \int_S (\overline{\mathbf{N}}_n \cdot \mathbf{u}_n + \overline{\mathbf{M}}_n \cdot \boldsymbol{\phi}_n)\,\mathrm{d}s - \int_D [\mathbf{N}_n \cdot (\mathbf{u}_n - \bar{\mathbf{u}}_n) + \mathbf{M}_n \cdot (\boldsymbol{\phi}_n - \bar{\boldsymbol{\phi}}_n)]\,\mathrm{d}s \tag{2}$$

and the equations of definition:

$$\boldsymbol{\varepsilon}_j = \frac{1}{\alpha_j \sin \Lambda}\left(\frac{\partial \mathbf{u}}{\partial x_j} + \frac{\partial \mathbf{r}}{\partial z_j} \times \boldsymbol{\phi}\right) \tag{3}$$

$$\boldsymbol{\kappa}_j = \frac{1}{\alpha_j \sin \Lambda}\frac{\partial \boldsymbol{\phi}}{\partial x_j}. \tag{4}$$

The upper bars in equation (2) denote prescribed values of the corresponding variables.

The Euler equations of equation (1) are:

(a) The equilibrium equations:

$$\frac{\partial}{\partial x_1}(\alpha_2 \mathbf{N}_1) + \frac{\partial}{\partial x_2}(\alpha_1 \mathbf{N}_2) + \alpha_1 \alpha_2 \mathbf{p} \sin \Lambda = 0 \tag{5}$$

$$\frac{\partial}{\partial x_1}(\alpha_2 \mathbf{M}_1) + \frac{\partial}{\partial x_2}(\alpha_1 \mathbf{M}_2) + \frac{\partial \mathbf{r}}{\partial x_1} \times (\alpha_2 \mathbf{N}_1) + \frac{\partial \mathbf{r}}{\partial x_2} \times (\alpha_1 \mathbf{N}_2) = 0. \tag{6}$$

(b) The stress strain relations:

$$\boldsymbol{\varepsilon}_i = \sum_{j=1}^{2} \frac{\partial W}{\partial N_{ij}} \mathbf{t}_j + \frac{\partial W}{\partial Q_i} \mathbf{n} \tag{7}$$

$$\boldsymbol{\kappa}_i = \sum_{j=1}^{2} \frac{\partial W}{\partial M_{ij}} \mathbf{t}_j + \chi_i \mathbf{n}\,*. \tag{8}$$

(c) The boundary conditions:

$$\mathbf{N}_n = \overline{\mathbf{N}}_n \tag{9}$$

and

$$\mathbf{M}_n = \overline{\mathbf{M}}_n \quad \text{on S} \tag{10}$$

$$\mathbf{u}_n = \overline{\mathbf{u}}_n \tag{11}$$

and

$$\boldsymbol{\phi}_n = \bar{\boldsymbol{\phi}}_n \quad \text{on D}. \tag{12}$$

All stress variables and displacements appearing in equation (2) can be varied freely whereas strain variables must be varied in such a way that equations (3) and (4) also hold for the variations of the variables involved.

Consider now a component representation (Fig. 1) such that the dot products $\mathbf{N}_i \, . \, \boldsymbol{\varepsilon}_i$ and $\mathbf{M}_i \, . \, \boldsymbol{\kappa}_i$ keep the same form as in orthogonal curvilinear coordinates when expressed in terms of the vector components:

$$\mathbf{N}_i = N_{ij}\mathbf{t}_j + Q_i \mathbf{n} \tag{13}$$

$$\boldsymbol{\kappa}_i = \kappa_{ij}\mathbf{t}_j + \chi_i \mathbf{n}. \tag{14}$$

$$\varepsilon_{ij} = \boldsymbol{\varepsilon}_i \, . \, \mathbf{t}_j; \qquad \gamma_i = \boldsymbol{\varepsilon}_i \, . \, \mathbf{n} \tag{15}$$

$$M_{ij} = \mathbf{M}_i \, . \, \mathbf{t}_j; \qquad \mathbf{M}_i \, . \, \mathbf{n} = 0. \tag{16}$$

Imagine now that equations (6), (7), (10), and (11) are identically satisfied by expressing N_{ij} in terms of $\mathbf{u}$ through equations (3) and (7), Q_i in terms of $\mathbf{M}_i$ through equation (6), and the boundary conditions equations (10) and (11) are directly enforced. It is then possible to

* Note that the component of $\boldsymbol{\kappa}_i$ along the normal to the shell, χ_i, does not have an associated stress-couple component $\mathbf{M}_i \, . \, \mathbf{n}$, so it must be obtained in terms of $\boldsymbol{\phi}$ from equation (4).

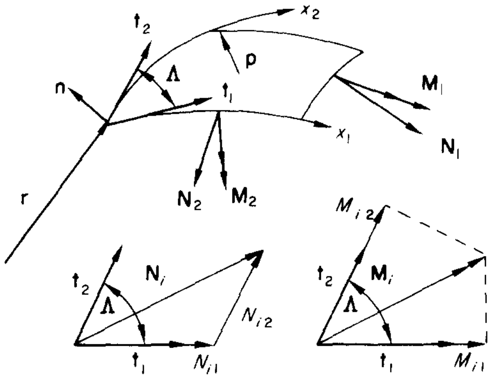

FIG. 1. Shell coordinates and component representation.

obtain the contracted form of Reissner's functional after a number of integrations by parts

$$I = \iint (W_M - W_B + U + (N_{12} - N_{21})\omega + Q_1(\gamma_1 - \beta_1) + Q_2(\gamma_2 - \beta_2))\,dA$$

$$- \int_S \bar{\mathbf{N}}_n \cdot \mathbf{u}_n\,ds + \int_D \mathbf{M}_n \cdot \bar{\boldsymbol{\phi}}_n\,ds \tag{17}$$

where

W_M = strain energy density associated with the in-plane stress resultants as function of $\mathbf{u}$

W_B = complementary strain energy density associated with the bending and transverse shear stresses as function of $\mathbf{M}_i$.

$$\boldsymbol{\phi} = -\beta_2\mathbf{t}_1 + \beta_1\mathbf{t}_2 + \omega\mathbf{n} \tag{18}$$

$$\gamma_i = C_s Q_i \text{ as function of } \mathbf{M}_i \tag{19}$$

β_i, ω are functions of $\mathbf{u}$ and $\mathbf{M}_i$ from equation (3).

The Euler equations of the functional in equation (17) are equations (5), (8), (9), and (12) expressed in terms of $\mathbf{u}$ and $\mathbf{M}_i$. Since equations (6), (7), (10) and (11) are identically satisfied, the Euler equations of equation (17) represent a reduction of the complete system of the shell equations.

3. FINITE ELEMENT SOLUTION

To illustrate the solution of the shell equations by means of a finite element procedure using the functional in equation (17), the linear equations for shallow shells [4] in orthogonal curvilinear coordinates are presented in discrete form. Obviously, an entirely similar procedure can be developed for nonshallow shells [11] which results in somewhat more involved algebraic expressions and the reader is referred to ref. 11 for the treatment of this case.

For shallow shells equations (3) and (4) in scalar form can be written*

$$\varepsilon_{11} = u_{\cdot 1} + \frac{w}{R_{11}} \tag{20}$$

$$\varepsilon_{22} = v_{\cdot 2} + \frac{w}{R_{22}} \tag{21}$$

$$\varepsilon_{12} = \varepsilon_{21} = \tfrac{1}{2}(u_{\cdot 2} + v_{\cdot 1}) + \frac{w}{R_{12}} \tag{22}$$

$$\kappa_{11} = \beta_{1,1} = \gamma_{1,1} - w_{,11} \tag{23}$$

$$\kappa_{22} = \beta_{2,2} = \gamma_{2,2} - w_{,22} \tag{24}$$

$$\kappa_{12} = \beta_{2,1} = \gamma_{2,1} - w_{,21} \tag{25}$$

$$\kappa_{21} = \beta_{1,2} = \gamma_{1,2} - w_{,12} \tag{26}$$

$$\gamma_1 = \beta_1 + w_{,1} \tag{27}$$

$$\gamma_2 = \beta_2 + w_{,2} \tag{28}$$

$$\mathbf{u} = u\mathbf{t}_1 + v\mathbf{t}_2 + w\mathbf{n} \tag{29}$$

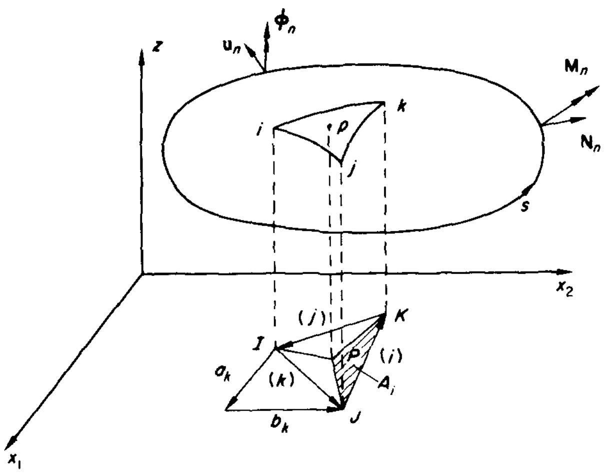

FIG. 2. Triangular coordinates in a shell element.

If it is assumed that $N_{12} = N_{21}$ and $M_{12} = M_{21}$ (the moment equilibrium equation about the normal will be violated by a negligible amount for thin shells [11]) W_M and W_B for an isotropic elastic thin shell can be taken as

$$W_M = \frac{Eh}{2(1-\nu^2)}[\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\nu\varepsilon_{11}\varepsilon_{22} + 2(1-\nu)\varepsilon_{12}\varepsilon_{21}] \tag{30}$$

* In order to conform with the usual notation for angular strains and stress couple components it is now taken: $\boldsymbol{\kappa}_i = -\kappa_{i2}\mathbf{t}_1 + \kappa_{i1}\mathbf{t}_2$ instead of that given by equation (14). Similarly $\mathbf{M}_i = -M_{i2}\mathbf{t}_1 + M_{i1}\mathbf{t}_2$ instead of that of equation (16).

$$W_B = \frac{6}{Eh^3}[M_{11}^2 + M_{22}^2 - 2\nu M_{11}M_{22} + 2(1+\nu)M_{12}M_{21}] + \frac{C_s}{2}[Q_1^2 + Q_2^2]. \tag{31}$$

Consider now a triangular shell element (Fig. 2) where a point P can be located by its triangular coordinates:

$$\zeta_i = \frac{A_i}{A} \tag{32}$$

where

$$A_i = \tfrac{1}{2}|\mathbf{JK} \times \mathbf{JP}|. \tag{33}$$

The following comparison functions are now defined for a point P in the element:

$$u = u_i\zeta_i \quad \text{sum on } i \tag{34}$$

$$v = v_i\zeta_i \tag{35}$$

$$w = w_i\zeta_i \tag{36}$$

$$M_{11} = M_{11_i}\zeta_i \tag{37}$$

$$M_{22} = M_{22_i}\zeta_i \tag{38}$$

$$M_{21} = M_{21_i}\zeta_i. \tag{39}$$

And the load intensity components

$$p_1 = p_{1_i}\zeta_i \tag{40}$$

$$p_2 = p_{2_i}\zeta_i \tag{41}$$

$$p_n = p_{n_i}\zeta_i. \tag{42}$$

The use of equations (34) to (39) in equations (20) to (22) and equations (30) and (31) and the introduction of:

$$Q_1 = M_{11,1} + M_{21,2} \tag{43}$$

$$Q_2 = M_{22,2} + M_{12,1} \tag{44}$$

together with equations (27) and (28) into equation (17), after taking the variations results in a system of algebraic linear equations for the nodal values of $\mathbf{u}$ and $\mathbf{M}_i$. Since one is interested in applying this variational principle to an assemblage of elements it is necessary to indicate what is the role played by the line integrals in equation (17). Consider now that all interelement boundaries are subject to continuity conditions of displacements and rotations. Thus, since M_{ij}'s are continuous across these boundaries, the line integrals along interior boundaries cancel out and only the integral along the actual boundary of the assemblage need be considered. This would not be the case if some components of $\mathbf{M}_n$ are discontinuous [7]. The boundary integral in equation (17) provides the natural boundary conditions for $\boldsymbol{\phi}_n$ and $\mathbf{N}_n$, variables which do not appear explicitly in this formulation. The boundary conditions for $\mathbf{u}_n$ and $\mathbf{M}_n$ are enforced by deleting the corresponding Euler equations at the boundary and replacing them with equations (11) and (10).

The system of equations for the complete structure is:

$$\frac{\partial I}{\partial u_i} = 0; \quad \frac{\partial I}{\partial v_i} = 0; \quad \frac{\partial I}{\partial w_i} = 0; \quad \frac{\partial I}{\partial M_{11_i}} = 0; \quad \frac{\partial I}{\partial M_{22_i}} = 0; \quad \frac{\partial I}{\partial M_{21_i}} = 0 \tag{45}$$

which may be written in matrix form as follows:

$$\mathbf{LV} = \mathbf{P} \tag{46}$$

where

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} & \dots & \mathbf{L}_{1n} \\ \vdots & & & \\ \mathbf{L}_{n1} & \dots & \dots & \mathbf{L}_{nn} \end{bmatrix} (6n \times 6n), \quad n = \text{number of nodes} \tag{47}$$

and

$$\mathbf{L}_{ij} = \begin{bmatrix} S^{ij}_{11} & S^{ij}_{12} & S^{ij}_{13} & 0 & 0 & 0 \\ S^{ij}_{21} & S^{ij}_{22} & S^{ij}_{23} & 0 & 0 & 0 \\ S^{ij}_{31} & S^{ij}_{32} & S^{ij}_{33} & C^{ij}_{11} & C^{ij}_{22} & C^{ij}_{21} \\ 0 & 0 & C^{ij}_{11} & F^{ij}_{11} & F^{ij}_{12} & F^{ij}_{13} \\ 0 & 0 & C^{ij}_{22} & F^{ij}_{21} & F^{ij}_{22} & F^{ij}_{23} \\ 0 & 0 & C^{ij}_{21} & F^{ij}_{31} & F^{ij}_{32} & F^{ij}_{33} \end{bmatrix} (6 \times 6) \tag{48}$$

where

$$S^{ij}_{11} = \frac{Eh}{4(1-v^2)A}\left[b_i b_j + \frac{1-v}{2} a_i a_j\right] \tag{49}$$

$$S^{ij}_{12} = \frac{-Eh}{4(1-v^2)A}\left[v b_j a_i + \frac{1-v}{2} a_j b_i\right] \tag{50}$$

$$S^{ij}_{13} = \frac{Eh}{6(1-v^2)}\left[-b_j\left(\frac{1}{R_{11}} + \frac{v}{R_{22}}\right) + a_j \frac{1-v}{R_{12}}\right] \tag{51}$$

$$S^{ij}_{21} = S^{ji}_{12} \tag{52}$$

$$S^{ij}_{22} = \frac{Eh}{4(1-v^2)A}\left[a_i a_j + \frac{1-v}{2} b_i b_j\right] \tag{53}$$

$$S^{ij}_{23} = \frac{Eh}{6(1-v^2)}\left[a_j\left(\frac{1}{R_{22}} + \frac{v}{R_{11}}\right) - b_j \frac{1-v}{R_{12}}\right] \tag{54}$$

$$S^{ij}_{31} = S^{ji}_{13} \tag{55}$$

$$S^{ij}_{32} = S^{ji}_{23} \tag{56}$$

$$S^{ij}_{33} = \begin{cases} \text{for } i = j: \quad = 2A\left[\dfrac{1}{R^2_{11}} + \dfrac{1}{R^2_{22}} + \dfrac{2v}{R_{11}R_{22}} + \dfrac{2(1-v)}{R^2_{12}}\right] & (57) \\ \text{for } i \neq j: \quad = A\left[\dfrac{1}{R^2_{11}} + \dfrac{1}{R^2_{22}} + \dfrac{2v}{R_{11}R_{22}} + \dfrac{2(1-v)}{R^2_{12}}\right] & (58) \end{cases}$$

$$C^{ij}_{11} = \frac{b_i b_j}{4A} \tag{59}$$

$$C^{ij}_{22} = \frac{a_i a_j}{4A} \tag{60}$$

$$C_{21}^{ij} = -\frac{a_i b_j + a_j b_i}{4A} \tag{61}$$

$$F_{11}^{ij} = \begin{cases} \text{for } i = j: & = -\left(\dfrac{2A}{Eh^3} + \dfrac{C_s b_j^2}{4A}\right) \quad (62) \\ \text{for } i \neq j: & = -\left(\dfrac{A}{Eh^3} + C_s \dfrac{b_i b_j}{4A}\right) \quad (63) \end{cases}$$

$$F_{12}^{ij} = \begin{cases} \text{for } i = j: & = \dfrac{2A\nu}{Eh^3} \quad (64) \\ \text{for } i \neq j: & = \dfrac{A\nu}{Eh^3} \quad (65) \end{cases}$$

$$F_{13}^{ij} = C_s \frac{a_i b_j}{4A} \tag{66}$$

$$F_{21}^{ij} = F_{12}^{ij} \tag{67}$$

$$F_{22}^{ij} = \begin{cases} \text{for } i = j: & = -\left(\dfrac{2A}{Eh^3} + \dfrac{C_s a_j^2}{4A}\right) \quad (68) \\ \text{for } i \neq j: & = -\left(\dfrac{A}{Eh^3} + C_s \dfrac{a_i a_j}{4A}\right) \quad (69) \end{cases}$$

$$F_{23}^{ij} = C_s \frac{a_j b_i}{4A} \tag{70}$$

$$F_{31}^{ij} = F_{13}^{ji} \tag{71}$$

$$F_{32}^{ij} = F_{23}^{ji} \tag{72}$$

$$F_{33}^{ij} = \begin{cases} \text{for } i = j: & = -\left[\dfrac{4A(1+\nu)}{Eh^3} + C_s \dfrac{(a_j^2 + b_j^2)}{4A}\right] \quad (73) \\ \text{for } i \neq j: & = -\left[\dfrac{2A(1+\nu)}{Eh^3} + C_s \dfrac{(a_i a_j + b_i b_j)}{4A}\right] \quad (74) \end{cases}$$

(a_j, b_j) are the Cartesian components of the oriented side "j" of the projection of the element into the x_1x_2 plane (Fig. 2)

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \vdots \\ \mathbf{V}_n \end{bmatrix}; \qquad \mathbf{V}_i = \begin{bmatrix} u_i \\ v_i \\ w_i \\ M_{11_i} \\ M_{22_i} \\ M_{21_i} \end{bmatrix} \tag{75}$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \vdots \\ \mathbf{P}_n \end{bmatrix}; \qquad \mathbf{P}_i = \begin{bmatrix} P_{1_i} \\ P_{2_i} \\ P_{3_i} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{76}$$

$$P_{1_i} = \frac{A}{6}\left(p_{1_i} + \sum_{j=1;\, j \neq i}^{3} \frac{p_{1_j}}{2}\right) \tag{77}$$

and similar expressions for P_{2_i} and P_{3_i}.

Consider now the triangular scheme of subdivision shown in continuous lines in Fig. 3. Clearly this pattern is biased in the sense that diagonals run in only one direction. It is found that without any significant increase in computational effort a quadrilateral element can be generated by averaging the coefficient matrix L as obtained for the patterns shown with continuous and dashed lines. It can be shown (example no. 1) that in this way a remarkable improvement in the accuracy and smoothness of the results is achieved without appreciably increasing the computational effort.

4. BOUNDARY CONDITIONS

The system of differential equations for shallow shells as derived from equation (17) and equations (20) to (28) is of tenth order, i.e. the solution of the shell boundary value problem requires that five boundary conditions are specified.

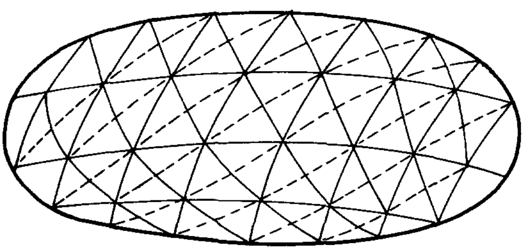

FIG. 3. Triangular patterns; quadrilateral element.

Let the boundary values be expressed in their component form by (Fig. 4)

$$\mathbf{u}_n = v_s\mathbf{t}_s + v_n\mathbf{t}_n + w\mathbf{n} \tag{78}$$

$$\mathbf{N}_n = N_{ns}\mathbf{t}_s + N_{nn}\mathbf{t}_n + Q_n\mathbf{n} \tag{79}$$

$$\boldsymbol{\phi}_n = -\beta_n\mathbf{t}_s + \beta_s\mathbf{t}_n \tag{80}$$

$$\mathbf{M}_n = -M_{nn}\mathbf{t}_s + M_{ns}\mathbf{t}_n. \tag{81}$$

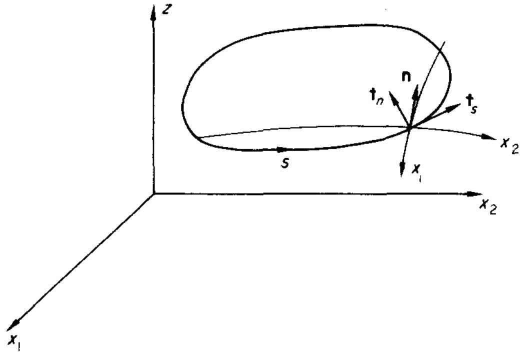

FIG. 4. Variables at the shell boundary.

At all points on the boundary three of the components in equations (78) and (79) must be prescribed. Clearly these three components chosen must be along the directions of $\mathbf{t}_s$, $\mathbf{t}_s$ and $\mathbf{n}$. Similarly two components in equations (80) and (81) must be prescribed. In order to specify the appropriate boundary conditions the components of $\mathbf{N}_n$ and $\mathbf{M}_n$ are transformed from the local reference system at the boundary ($\mathbf{t}_s$, $\mathbf{t}_n$, $\mathbf{n}$) to the global curvilinear system ($\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{n}$) according to vector and second order tensor transformations respectively.

Obviously, for the case of zero transverse shear strains, the component M_{ns} cannot be varied freely in equation (17) and a contraction of the boundary conditions as presented in Ref. [3] must be performed. In this way one arrives at a system of boundary conditions for an effective stress resultant $\mathbf{N}_{ne}$ and M_{nn} or the corresponding displacement quantities $\mathbf{u}_n$ and β_n. The expressions for these effective stress resultants are given in Ref. [3].

5. NUMERICAL EXAMPLES

A computer program has been implemented using the present formulation. Some selected examples are now presented.

Example 1, *Rectangular plate* (*Fig.* 5)

The plate has no transverse shear deformability. Nevertheless, it is observed that when this finite element procedure is used it is possible to impose five boundary conditions to the discrete form of the governing equations. In the case at hand, the variable M_{21} has a very steep variation in the vicinity of the free edge; at the edge itself M_{21} vanishes and increases very steeply as a function of x_2 inwards. Clearly this situation is not reflected by the series solution which satisfies the contracted form of the boundary conditions. Figure 6 presents

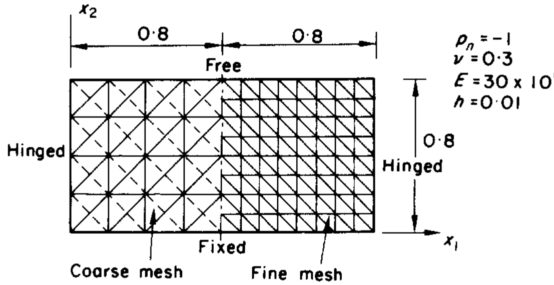

FIG. 5. Rectangular plate under normal pressure.

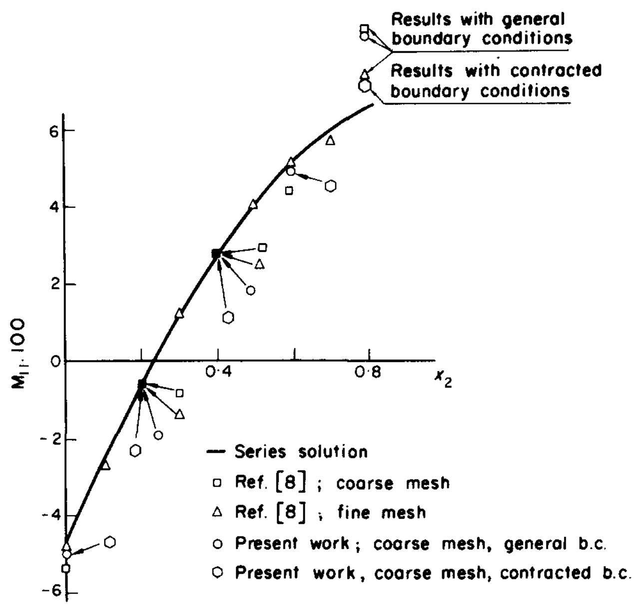

FIG. 6. Comparison of results for M_{11} along line $x_1 = 0·4$.

a comparison of the results given by the series solution, those of Ref. [8]. and the present ones for M_{11}. In order to illustrate the importance of specifying the contracted form of the boundary conditions, the values of M_{11} obtained by specifying five boundary conditions and those obtained with their contracted form are presented in Fig. 6. A more complete comparison of results is presented in Figs. 7–9.

Example 2, *Clamped spherical shell* (*Fig.* 10)

The present results are compared with an analytical solution [6] and with another finite element solution [9] in Figs. 10 and 11.

Example 3, *Pressurized cylindrical shell with a circular hole* (*Fig.* 12)

The edge of the hole is considered free from bending and in-plane stresses and subject to a uniform transverse shear that equilibrates the internal pressure. A comparison of

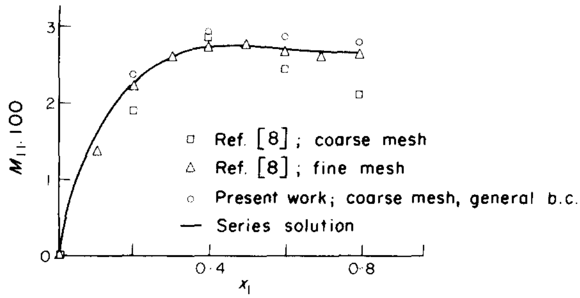

FIG. 7. Comparison of results for M_{11} along line $x_2 = 0{\cdot}4$.

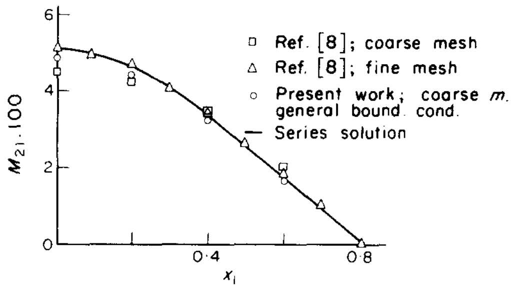

FIG. 8. Comparison of results for M_{21} along line $x_2 = 0{\cdot}4$.

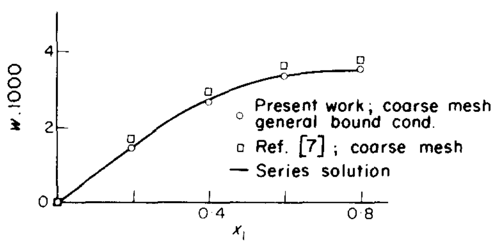

FIG. 9. Comparison of results for w along line $x_1 = 0{\cdot}4$.

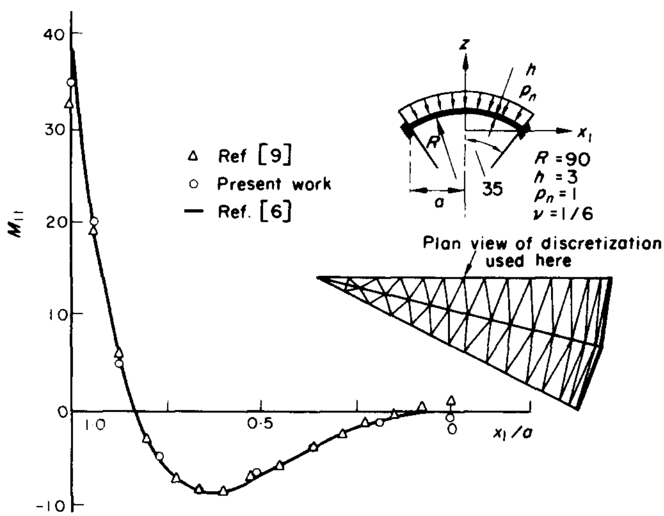

FIG. 10. Spherical shell under uniform normal pressure. Comparison of results for M_{11} along meridian $x_2 = 0$.

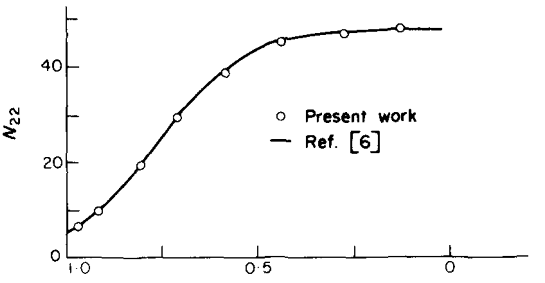

FIG. 11. Comparison of results for N_{22} along meridian $x_2 = 0$.

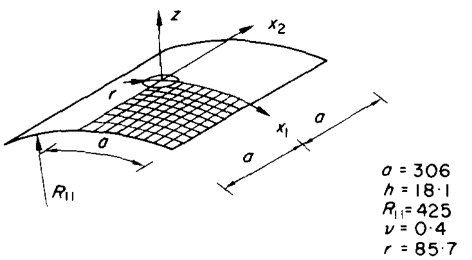

FIG. 12. Pressurized cylindrical shell with circular hole.

stresses at the outside and inside surfaces with another approximate solution [10] is presented in Figs. 13 and 14.

6. CONCLUSIONS

A specialized form of Reissner's Principle suitable for use in conjunction with the finite element procedure for thin elastic shell has been developed. The stress couples and linear displacements are the dependent variables of the formulation.

The numerical results indicate that the accuracy of this finite element procedure compares favorably with previous finite element methods. This mixed method has an advantage

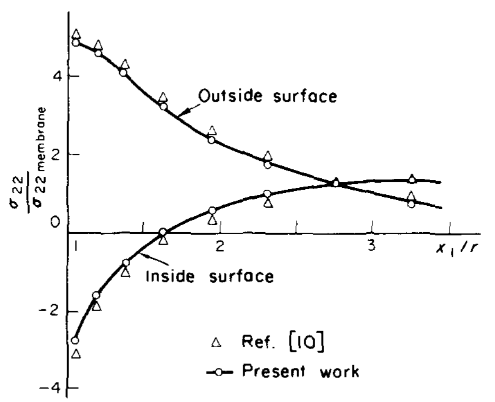

FIG. 13. Comparison of results for σ_{22} along x_1 axis.

over the displacement method in that the resulting discrete equations are considerably simpler. It is also observed that the consideration of transverse shear strains does not appreciably increase the computational effort.

The finite element procedure presented here implies bending stresses which are continuous across the shell whereas the in-plane stress resultants are not. An obvious refinement to the present procedure would be to use polynominal expansions for the displacement components u and v of a degree higher than the first.

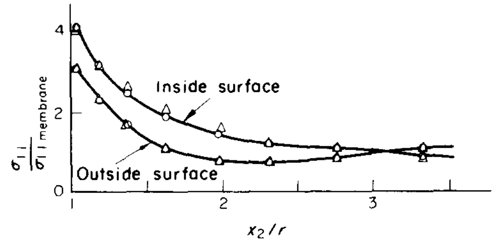

FIG. 14. Comparison of results for σ_{11} along x_2 axis.

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Абстракт—Предлагается специализированная форма вариационного принципа Рейсснера для напряжений и перемещений, пригодная для формулировки варианта конечного элемента определяющих уравнений в теории тонких оболочек. Уравнения Эилера для модифицировнного функционала Рейсснера, являются трема уравнениями равновесия сил и трема моментными зависимостями кривизны, выраженными в виде уравнений равновесия линейного вектора смещения и сопряженных векторов напряжений. Получаются граничные условия в форме очевидных условий этого функционала. Область оболочки разлагается на совокупность трехугольных или четырехугольных элементов, с целью определения системы кусочных линейных функций, как сравнительных функций, для зависимых переменных этой формулировки. Использование этих функций, выраженных членами их значений в узлах, в вариационном принципе приводит к системе алгебраических, совместных уравнений, для узловых переменных. Особенностью этого метода, по сравнению с другими приближенными методами решения, оказывается его алгебраическая простота и точность. Некторые численные примеры иллюстрируют практически особенности метода. Когда этот метод применяется специально для анализа изгибаемой пластинки, наблюдается значительное уточнение, касающееся предыдущей пзвестной методики конечного элемента.